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(YIP) Multilinear Computing and Multilinear Algebraic Geometry

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**08/10/2016**  
**Final Report**

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<b>14. ABSTRACT</b> The following report details various aspects of this project. The main narrative in Section 1 describes the highlights of the most significant achievements as well as potential future developments. The next few sections contain a list of invited talks, published papers, submitted preprints, conferences organized, and personnel supported during the performance period of this project.						
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# Final Performance Report

Lek-Heng Lim

March 31, 2016

**Grant Title:** (YIP) Multilinear Computing and Multilinear Algebraic Geometry **Grant**

**Number:** FA9550-13-1-0133

**Performance Period:** March 27, 2013–March 31, 2016

## 1 Comprehensive summary of significant work accomplished

We will only be able to provide a highlight of selected works to avoid an unduly long report. We refer the reader to the cited papers in Sections 3 and 4 for the details. Support from this grant has been explicitly acknowledged in all the following works.

The first and second items below were reported in our 2013/14 annual report; the third and fourth item were reported in our 2014/15 annual report. However all these works have seen further developments during the later stages of our performance period. In particular, there are now several more publications that resulted from these subsequent developments.

**NP-hardness of tensor problems:** With his collaborator Chris Hillar of Berkeley, the PI published a landmark paper titled “Most tensor problems are NP-hard” (see [14] in Section 3) in the *Journal of the ACM*, the premier journal in Computer Science. The results in this paper is a validation of this project. We showed that just about every multilinear generalization of common place problems in linear algebra: linear systems, least squares, eigvalue problems, singular value problems, determinant evaluation, low-rank approximations, etc — problems that form the bedrock of scientific and engineering computing — are all intractable. The implication is that one will have to find a way around this intractability to move beyond linear algebra, substantiating what the PI had proposed.

**High-resolution MRI with tensors:** In another piece of work with Thomas Schultz et al. (see [4, 13] in Section 3), we showed how tensor-based techniques can be used to reconstruct, to very high-resolutions, images obtained from diffusion MRI methods.

**Multilinear PageRank:** With David Gleich and Yongyang Yu from Purdue, as well as Austin Benson from Stanford, we extended the celebrated PageRank algorithm to a multilinear setting in which the “random surfer” model is replaced with a more realistic “spacey random surfer.” We show that in place of a simple Markov chain, the random process behind this is the vertex reinforced random walk. This is a great example of a situation where we can

produce an approximation to a multilinear problem that makes it computationally tractable. This work has been published as [9] in Section 3 and [17] in Section 4.

**Nuclear norms of tensors:** There has been an enormous amount of interest and work purporting to extend compressive sensing/sparse recovery (for vectors) and matrix completion (for matrices) to tensors. The PI has shown that this is impossible. The reason is as follows. Compressed sensing relies crucially on the fact that the  $l^1$ -norm is a convex underestimator of the “ $l^0$ -norm” on the  $l^\infty$ -norm unit ball; matrix completion relies crucially on the fact that the nuclear norm is a convex underestimator of rank on the spectral norm unit ball. The PI has shown in a paper with Pierre Comon of Grenoble (see [12] in Section 3) that the generalization of this statement is *false* for tensors of order 3 or higher. The PI would also like to highlight that, in more recent work with Shmuel Friedland, we have thoroughly investigated the properties of tensor nuclear norm, going far beyond [12]. This new work is available as the preprint [18] in Section 4.

**Complexity of duality:** In joint work with Shmuel Friedland of UIC, the PI showed that for any given norm ball or proper cone, weak membership in its dual ball or dual cone is polynomial-time reducible to weak membership in the given ball or cone. A consequence is that the weak membership or membership problem for a ball or cone is NP-hard if and only if the corresponding problem for the dual ball or cone is NP-hard. In a similar vein, we show that computation of the dual norm of a given norm is polynomial-time reducible to computation of the given norm. This extends to convex functions satisfying a polynomial growth condition: for such a given function, computation of its Fenchel dual/conjugate is polynomial-time reducible to computation of the given function. Hence the computation of a norm or a convex function of polynomial growth is NP-hard if and only if the computation of its dual norm or Fenchel dual is NP-hard. We discuss implications of these results on the weak membership problem for a symmetric convex body and its polar dual, the polynomial approximability of Mahler volume, and the weak membership problem for the epigraph of a convex function with polynomial growth and that of its Fenchel dual. This paper has been submitted and is available as a preprint (see [19] in Section 4)

**Semialgebraic geometry of nonnegative tensor rank:** This is joint work with Yang Qi, a post-doctoral scholar jointly supervised by Pierre Comon and the PI. We study the semialgebraic structure of  $D_r$ , the set of nonnegative tensors of nonnegative rank not more than  $r$ , and use the results to infer various properties of nonnegative tensor rank. We determine all nonnegative typical ranks for cubical nonnegative tensors and show that the direct sum conjecture is true for nonnegative tensor rank. Under some mild condition (non-defectivity), we show that nonnegative, real, and complex ranks are all equal for a general nonnegative tensor of nonnegative rank strictly less than the complex generic rank. In addition, such nonnegative tensors always have unique nonnegative rank- $r$  decompositions if the real tensor space is  $r$ -identifiable. We determine conditions under which a best nonnegative rank- $r$  approximation has a unique nonnegative rank- $r$  decomposition: for  $r \leq 3$ , this is always the case; for general  $r$ , this is the case when the best nonnegative rank- $r$  approximation does not lie on the boundary of  $D_r$ . This work has been published as [6] but builds on our earlier work [8] in Section 3.

As is often the case in research work, the pursuit of the original goals could lead to unanticipated developments. There have been two lines of such developments that evolved naturally from the work in this project. Both developments are in-line with the scope of our original proposal but it will require significant further investments (in terms of time, energy, and funding) to develop them further — the PI plans to apply again to AFOSR as well as other funding agencies to support the following works but would like to report some initial findings in the next two subsections.

## 1.1 Distances between subspaces of different dimensions

The study of Grassmann varieties (described in Section 4 of our original proposal) have led to the following development: The Grassmann variety parameterizes all linear subspaces of a given dimension in an ambient space. A distance measure on this variety naturally gives rise to a distance between any two linear subspaces of the same dimension. What about linear subspaces of different dimensions? We have discovered a notion of distance based on Schubert varieties that allows us to extend any distance measure for linear subspaces of the same dimension to linear subspaces of different dimensions. Furthermore such distances can be computed stably and efficiently in polynomial-time. This has been described in the paper [5] in Section 3.

There is more: The next stage of this work involves Dr. Ke Ye, the PI’s postdoc working on this project and Ken Sze-Wai Wong, the PI’s PhD student (see Section 6). This next stage is firstly an extension of the above to *affine* subspaces of different dimensions. But while developing the idea, we realized that the affine Grassmannian – the analogue of the Grassmann variety for affine subspaces — is relatively unstudied and we have decided to lay the foundations for this object (defining distances, probability densities, developing optimization algorithms, and studying other basic properties, etc). We expect the affine Grassmannian to be important in practical engineering applications. One reason is that we found out that many statistical estimation problems (linear regression, errors-in-variables regression, principal components analysis, support vector machines, etc) can be formulated as optimization problems over the affine Grassmannian. This is described in detail in the preprint [20] in Section 4.

## 1.2 Fastest algorithms for structured matrix computations

The study of tensor rank and tensor decompositions (described in Section 5.2 of our original proposal) have led to the following development: While finding interesting tensors to investigate, we realized that an important class of tensors are the “structure tensors” of common bilinear operations in numerical computing. Decompositions of such tensors give us explicit algorithms to compute such bilinear operations. Our initial study has now appeared in the papers [1, 3] in Section 3, where we determine the tensor rank decompositions of many structure tensors arising from the most common bilinear operations in use, particularly for the structured matrix-vector products with various structured matrices (Toeplitz, Hankel, circulant, Toeplitz-plus-Hankel, BTTB, banded, symmetric, etc). The next step of this work is to translate all these decompositions into actual algorithms that can be readily used by practitioners. By what we have shown, these algorithms will be the fastest possible in terms of bilinear complexity.

## 2 Invited talks during performance period

The PI gave the following invited talks during the performance period. This grant is explicitly acknowledged in the slides used for the presentations.

- “Hypermatrixes,” Mathematics Colloquium, Department of Mathematics, University of California, San Diego, CA, March 3, 2016.
- “Blind multilinear identification and tensor nuclear norm,” Workshop on Low Complexity Models in Signal Processing, Hausdorff Center for Mathematics, Bonn, Germany, February 15–19, 2016.
- “Grothendieck constant and structured matrix computations,” Workshop Optimization and Parsimonious Modeling, Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, MN, January 25–29, 2016.
- “Grothendieck constant, quantum entanglement, and numerical stability,” Workshop on Tensor Decompositions and Applications, Leuven, Belgium, January 18–22, 2016.
- “Learning subspaces of different dimensions,” Conference of the International Association for Statistical Computing–Asia Regional Section, Singapore, December 17–19, 2015.
- “Higher-order cone programming,” Machine Learning Thematic Trimester, International Centre for Mathematics and Computer Science, Toulouse, France, October 6–7, 2015.
- “Fast randomized iteration for matrix inversion, eigenproblems, and exponentiation,” Minisymposium on Numerical Linear Algebra Techniques in Massive Data Analysis, International Congress on Industrial and Applied Mathematics, Beijing, China, August 10–14, 2015.
- “Structured matrix computations via tensor decompositions,” Workshop in Low-Rank Optimization Techniques, Hausdorff Center for Mathematics, Bonn, Germany, June 8–12, 2015.
- “Fast(est) algorithms for structured matrices via tensor decompositions,” Applied Mathematics and Analysis Seminar, Duke University, Durham, NC, April 13, 2015.
- “Fast(est) algorithms for structured matrices via tensor decompositions,” Applied Mathematics Seminar, Stanford University, Stanford, CA, March 11, 2015.
- “The role of tensors in numerical mathematics,” Winter School on Search for Latent Variables: ICA, Tensors, and NMF, Grenoble, France, February 2–4, 2015.
- “Schubert varieties and distances between subspaces of different dimensions,” Workshop on Computational Algebraic Geometry, Foundations of Computational Mathematics Conference, Montevideo, Uruguay, December 18–20, 2014.
- “Factoring arbitrary matrices into product of structured matrices,” Workshop on Numerical Linear Algebra, Foundations of Computational Mathematics Conference, Montevideo, Uruguay, December 18–20, 2014.

- “Factoring arbitrary matrices into product of structured matrices,” Structured Matrices and Tensors: Analysis, Algorithms and Applications, National Taiwan University, Taipei, Taiwan, December 8–11, 2014.
- “Distances between subspaces of different dimensions,” Scientific Computing and Matrix Computations Seminar, University of California, Berkeley, CA, November 19, 2014.
- “Connection Laplacian, Hodge Laplacian, and tensor Laplacian of a graph,” Workshop on Spectral Algorithms: From Theory to Practice, Simons Institute for the Theory of Computing, Berkeley, CA, October 27–31, 2014.
- “Learning subspaces of different dimensions,” Workshop on Mathematical Issues in Information Sciences, Xi’an, China, July 5–10, 2014.
- “Blind multilinear identification,” Department of Mathematics, Tsinghua University, Beijing, China, June 19, 2014.
- “Moment tensors and high angular resolution diffusion imaging,” Minisymposium on Image Processing for Tensor- and Manifold-Valued Data, SIAM Conference on Imaging Science, Hong Kong, May 12–14, 2014.
- “Hypermatrixes,” Mathematics Colloquium, Department of Mathematics, University of Wisconsin, Madison, WI, May 2, 2014.
- “Hypermatrixes,” Mathematics Colloquium, Department of Mathematics, University of Illinois, Urbana-Champaign, IL, April 24, 2014.
- “Learning subspaces of different dimensions,” Statistics Colloquium, Department of Statistics, Pennsylvania State University, University Park, PA, April 17, 2014.
- “Hypermatrixes,” Colloquium, Department of Mathematics and Statistics, McMaster University, Hamilton, ON, April 4, 2014.
- “Subspace distances,” Symposium on Hot Topics at the Interface of Mathematics and Engineering, Initiative for Mathematical Sciences and Engineering, University of Illinois, Urbana-Champaign, IL, February 24–25, 2014.
- “Hodge decomposition in data analysis,” Program on Low Dimensional Structure in High Dimensional Systems, Statistical and Applied Mathematical Sciences Institute, Research Triangle Park, NC, February 3–7, 2014.
- “The role of tensors in numerical mathematics,” Peaceman Lecture, Department of Computational and Applied Mathematics, Rice University, Houston, TX, January 27, 2014.
- “Hodge decomposition in data analysis,” AMS Short Course on Geometry and Topology in Statistical Inference, Joint Mathematics Meetings, Baltimore, MD, January 13, 2014.
- “Tensors in optimization theory,” International Conference on Engineering and Computational Mathematics, Hong Kong, December 16–18, 2013.



- “Numerical linear algebra for subspace learning,” TIMS Winter School for Scientific Computing, Taida Institute of Mathematical Sciences, National Taiwan University, Taipei, Taiwan, December 5, 2013.
- “Tensors and hypermatrices,” Mathematics Colloquium, Department of Mathematics, University of Alabama, Birmingham, AL, September 27, 2013.
- “Symmetric tensors with positive decompositions,” Minisymposium on Tensor Methods in Mathematics and Physics, ENUMATH Conference, École Polytechnique Fédérale de Lausanne, Switzerland, August 26–30, 2013.
- “Blind multilinear identification,” Minisymposium on Algebro-geometric Approaches to Tensor Spaces, Tensor Decomposition, and Identifiability, 2013 SIAM Conference on Applied Algebraic Geometry, Fort Collins, CO, August 1–4, 2013.
- “Differentiable, continuous, and combinatorial Hodge theories,” Minisymposium on Algebraic Aspects of Large-Scale Statistics, 2013 SIAM Conference on Applied Algebraic Geometry, Fort Collins, CO, August 1–4, 2013.
- “Lectures on tensors and hypermatrices,” (4 lectures), Centro de Estruturas Lineares e Combinatórias, University of Lisbon, Lisbon, Portugal, July 22–26, 2013.
- “Algebraic geometry of matrices,” (4 lectures), Advanced School and Workshop on Matrix Geometries and Applications, Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, July 1–12, 2013.
- “Tensors and hypermatrices,” 100th Anniversary of the School of Mathematics: Young Mathematician Forum, Peking University, Beijing, China, June 18–21, 2013.
- “Multiplying higher-order tensors,” Session on Multilinear Algebra and Tensor Decompositions, 18th ILAS Conference, Providence, RI, June 3–7, 2013.
- “Complexity and symmetry,” Special Session on Computability and Complexity in Discrete and Continuous Worlds, AMS Spring Central Sectional Meeting, Ames, IA, April 27–28, 2013.
- “Hypermatrixes,” Departmental Colloquium, Department of Mathematics, University of Illinois, Chicago, IL, April 26, 2013.
- “Hypermatrixes,” Center for Computational and Applied Mathematics Seminar, Department of Mathematics, Purdue University, West Lafayette, IN, April 5, 2013.

### 3 Publications during performance period

The PI published the following papers during this period. This grant is explicitly acknowledged in all of them.

- [1] K. Ye and L.-H. Lim, “Fast structured matrix computations: tensor rank and Cohn–Umans method,” *Found. Comput. Math.*, (2016), to appear.
- [2] L.-H. Lim and J. Weare, “Fast randomized iteration: diffusion Monte Carlo through the lens of numerical linear algebra,” *SIAM Rev.*, (2016), to appear.
- [3] K. Ye and L.-H. Lim, “Algorithms for structured matrix-vector product of optimal bilinear complexity,” *Proc. IEEE Inform. Theory Workshop (ITW)*, **16** (2016), to appear.
- [4] M. Ankele, L.-H. Lim, S. Groeschel, and T. Schultz, “Fast and accurate multi-tissue deconvolution using SHORE and H-PSD tensors,” *Proc. Internat. Conf. Med. Image Comput. Comput. Assist. Interv. (MICCAI)*, **19** (2016), to appear.
- [5] K. Ye and L.-H. Lim, “Schubert varieties and distances between subspaces of different dimensions,” *SIAM J. Matrix Anal. Appl.*, (2016), to appear.
- [6] Y. Qi, P. Comon, and L.-H. Lim, “Semialgebraic geometry of nonnegative tensor rank,” *SIAM J. Matrix Anal. Appl.*, (2016), to appear.
- [7] K. Ye and L.-H. Lim, “Every matrix is a product of Toeplitz matrices,” *Found. Comput. Math.*, **16** (2016), no. 3, pp. 577–598.
- [8] Y. Qi, P. Comon, and L.-H. Lim, “Uniqueness of nonnegative tensor approximations,” *IEEE Trans. Inform. Theory*, **62** (2016), no. 4, pp. 2170–2183.
- [9] D. Gleich, L.-H. Lim, and Y. Yu, “Multilinear PageRank” *SIAM J. Matrix Anal. Appl.*, **36** (2015), no. 4, pp. 1507–1541.
- [10] L.-H. Lim, “Hodge Laplacians on graphs,” S. Mukherjee (Ed.), *Geometry and Topology in Statistical Inference, Proc. Sympos. Appl. Math.*, **73**, AMS, Providence, RI, 2015.
- [11] A. Rajkumar, S. Ghoshal, L.-H. Lim, and S. Agarwal, “Ranking from stochastic pairwise preferences: recovering Condorcet winners and tournament solution sets at the top,” *Proc. Int. Conf. Mach. Learn. (ICML)*, **37** (2015), pp. 665–673.
- [12] L.-H. Lim and P. Comon, “Blind multilinear identification,” *IEEE Trans. Inform. Theory*, **60** (2014), no. 2, pp. 1260–1280.
- [13] T. Schultz, A. Fuster, A. Ghosh, L. Florack, R. Deriche, and L.-H. Lim, “Higher-order tensors in diffusion imaging,” pp. 129–161, C.-F. Westin et al. (Eds.), *Visualization and Processing of Tensors and Higher Order Descriptors for Multi-Valued Data*, Springer-Verlag, Berlin Heidelberg, 2014.
- [14] C. J. Hillar and L.-H. Lim, “Most tensor problems are NP-hard,” *J. ACM*, **60** (2013), no. 6, Art. 45, 39 pp.
- [15] L.-H. Lim, “Tensors and hypermatrices,” Art. 15, 30 pp., in L. Hogben (Ed.), *Handbook of Linear Algebra*, 2nd Ed., CRC Press, Boca Raton, FL, 2013.

- [16] M. Gu, L.-H. Lim, and C. Wu, “PARNES: A rapidly convergent algorithm for accurate recovery of sparse and approximately sparse signals,” *Numer. Algorithms*, **64** (2013), no. 2, pp. 321–347.

## 4 Preprints submitted

The following preprints contain work done during this period. This grant is explicitly acknowledged in all of them.

- [17] A. Benson, D. Gleich, and L.-H. Lim, “The spacey random walk: a stochastic process for higher-order data,” <http://arxiv.org/abs/1602.02102>.
- [18] S. Friedland and L.-H. Lim, “Nuclear norm of higher-order tensors,” <http://arxiv.org/abs/1410.6072>.
- [19] S. Friedland and L.-H. Lim, “The computational complexity of duality,” <http://arxiv.org/abs/1601.07629>.
- [20] L.-H. Lim, K. S.-W. Wong, and K. Ye, “Statistical estimation and the affine Grassmannian,” <http://arxiv.org/abs/1607.01833>.
- [21] L.-H. Lim, “Self-concordance is NP-hard,” <http://arxiv.org/abs/1303.7455>.
- [22] B. St. Thomas, L. Lin, L.-H. Lim, and S. Mukherjee, “Learning subspaces of different dimensions,” <http://arxiv.org/abs/1404.6841>.
- [23] K. Ye and L.-H. Lim, “Cohomology of cyro-electron microscopy,” <http://arxiv.org/abs/1604.01319>.

## 5 Conference organized during performance period

The PI organized the following professional meetings in part to seek interactions and potential collaborations on various aspects of the work related to this project.

- 2015 STEVANOVICH CENTER CONFERENCES: Conference on *Geometry and Data Analysis*, University of Chicago, IL, Jun 8–10, 2015 (with S. Weinberger), <http://stevanovichcenter.uchicago.edu/page/conference-geometry-and-data-analysis>
- 2014 SIMONS INSTITUTE WORKSHOP: Workshop on *Tensors in Computer Science and Geometry*, University of California, Berkeley, CA, Fall 2014 (with H. Cohn, J. Morton, G. Ottaviani), [http://simons.berkeley.edu/workshop\\_alggeometry3.html](http://simons.berkeley.edu/workshop_alggeometry3.html)
- 2014 IMA SUMMER GRADUATE PROGRAM: Summer School on *Modern Applications of Representation Theory*, University of Chicago, Chicago, IL, Summer 2014 (with R. Kondor, J. Morton), <http://www.ima.umn.edu/2013-2014/PISG/>

- 2014 INTERNATIONAL CONFERENCE ON MACHINE LEARNING: ICML Workshop on *Topological Methods for Machine Learning*, Beijing, China, Jun 25–26, 2014 (with Y. Yao, J. Zhu, X. Zhu), <http://topology.cs.wisc.edu/>
- 2014 ICERM TOPICAL WORKSHOP: Workshop on *Computational Nonlinear Algebra*, Brown University, Providence, RI, Jun 2–6, 2014 (with G. Blekherman, P. Parrilo, A. Sommesse, R. Thomas), <http://icerm.brown.edu/tw-14-3-cna>
- 2014 SIAM CONFERENCE ON OPTIMIZATION: Minisymposium on *Tensor and Optimization Problems*, San Diego, CA, May 22, 2014 (with J. Nie), [http://meetings.siam.org/session/dsp\\_programsess.cfm?SESSIONCODE=18377](http://meetings.siam.org/session/dsp_programsess.cfm?SESSIONCODE=18377)
- 2013 IMA ANNUAL PROGRAM’: Workshop on *Modern Applications of Homology and Cohomology*, University of Minnesota, Minneapolis, MN, Oct 28–Nov 1, 2013 (with A. Blumberg, Y. Yao), <http://www.ima.umn.edu/2013-2014/W10.28-11.1.13/>
- 2013 SIAM CONFERENCE ON APPLIED ALGEBRAIC GEOMETRY: Minisymposium on *Algebraic Geometry of Tensor Decompositions*, Fort Collins, CO, Aug 1–4, 2013, [http://meetings.siam.org/session/dsp\\_programsess.cfm?SESSIONCODE=16740](http://meetings.siam.org/session/dsp_programsess.cfm?SESSIONCODE=16740)
- 2013 INTERNATIONAL CONFERENCE ON CONTINUOUS OPTIMIZATION: Session on *Algebraic Geometry and Semidefinite Programming*, Lisbon, Portugal, Jul 27–Aug 1, 2013 (with C. Reiner), <http://www.norg.uminho.pt/iccopt2013-SP-contents/cluster3.htm>

## 6 Personnel involved

The following is a list of all personnel who have worked with the PI during the performance period of this project. Their involvements with the project range from tangential to substantial. Two postdoctoral scholars Shenglong Hu and Ke Ye were each funded for a few months by this grant. A few others had received some amount of travel support or computing equipment paid for from this grant. The remaining personnel were not directly supported from this grant but benefited in indirect ways like working under the PI’s supervision during summer months (when the PI is paid summer salary from this grant).

### 6.1 Postdocs

- SHENGLONG HU, Postdoctoral Scholar, 2015–2016, now Associate Professor in Mathematics at Tianjin University
- YANG QI, Postdoctoral Scholar, joint with Pierre Comon, since 2013
- JOSE RODRIGUEZ, NSF Postdoctoral Fellow and Provost’s Postdoctoral Scholar since 2015, winner of 2016 Provost’s Career Enhancement Postdoctoral Scholarship
- ANNE SHIU, Dickson Instructor and NSF Postdoctoral Fellow, joint with Mathias Drton, 2011–2014, now Assistant Professor in Mathematics at Texas A&M University

- KE YE, Dickson Instructor and Postdoctoral Scholar, since 2012

## 6.2 PhD Students

- GREGORY NAITZAT, Department of Statistics, since 2015
- KATHERINE TURNER, Department of Mathematics, joint with Shmuel Weinberger, 2010–2015, winner of Nadine Kowalsky Prize, now Postdoc in Statistics at EPFL
- KEN SZE-WAI WONG, Department of Statistics, since 2012
- LIWEN ZHANG, Department of Computer Science, since 2011

## 6.3 MS Students

- FREDDY BOULTON, Department of Statistics, 2015–2016, undergraduate in BS/MS program
- LIJUN DING, Department of Statistics, 2014–2016, now PhD student in Operations Research and Industrial Engineering, Cornell University
- BYOL KIM, Department of Statistics, 2013–2015, now PhD student in Statistics, University of Chicago
- SHAN LU, Department of Statistics, joint with Sou-Cheng Choi, since 2015
- JOHN SANTERRE, Department of Computer Science, 2012–2014, now PhD student in Computer Science, University of Chicago

## 7 Thanks

The PI would like to express his sincere gratitude to AFOSR and his program managers for their kind support. This award has been instrumental in furthering not just the PI's career but also those of many junior personnel involved (see Section 6). Thank you very much for this opportunity.

1.

**1. Report Type**

Final Report

**Primary Contact E-mail****Contact email if there is a problem with the report.**

lekheng@galton.uchicago.edu

**Primary Contact Phone Number****Contact phone number if there is a problem with the report**

650-387-6896

**Organization / Institution name**

University of Chicago

**Grant/Contract Title****The full title of the funded effort.**

Multilinear Computing and Multilinear Algebraic Geometry

**Grant/Contract Number****AFOSR assigned control number. It must begin with "FA9550" or "F49620" or "FA2386".**

FA9550-13-1-0133

**Principal Investigator Name****The full name of the principal investigator on the grant or contract.**

Lek-Heng Lim

**Program Manager****The AFOSR Program Manager currently assigned to the award**

Doug Riecken

**Reporting Period Start Date**

03/27/2013

**Reporting Period End Date**

03/31/2016

**Abstract**

With Shmuel Friedland of UIC, we proved in [19] that a norm is NP-hard to compute iff its dual norm is NP-hard to compute, and extended such computational equivalence to a number of dual objects commonly occurring in optimization. In [18] we showed how one may naturally extend the matrix nuclear norm to higher-order tensors and view it as a continuous variant of tensor rank. We established several surprising properties: tensor nuclear norm depends on the choice of base field (like tensor rank), is upper semicontinuous (unlike tensor rank), satisfies the Comon's conjecture (unknown for tensor rank), and is NP-hard to compute (applying the result in [19]).

In joint work with Pierre Comon and the PI's postdoc Yang Qi, we investigated a long list of properties of nonnegative tensor rank that is, in our view, as exhaustive as current technologies allow. See [6, 8] for more information.

**Distribution Statement****This is block 12 on the SF298 form.**

Distribution A - Approved for Public Release

**Explanation for Distribution Statement**

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[final2016.pdf](#)

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### Archival Publications (published) during reporting period:

K. Ye and L.-H. Lim, "Fast structured matrix computations: tensor rank and Cohn–Umans method," Foundations of Computational Mathematics, (2016), to appear.

L.-H. Lim and J. Weare, "Fast randomized iteration: diffusion Monte Carlo through the lens of numerical linear algebra," SIAM Review, (2016), to appear.

Y. Qi, P. Comon, and L.-H. Lim, "Semialgebraic geometry of nonnegative tensor rank," SIAM Journal on Matrix Analysis and Applications, (2016), to appear.

K. Ye and L.-H. Lim, "Algorithms for structured matrix-vector product of optimal bilinear complexity," Proceedings of the IEEE Information Theory Workshop (ITW), (2016), to appear.

Y. Qi, P. Comon, and L.-H. Lim, "Uniqueness of nonnegative tensor approximations," IEEE Transactions on Information Theory, 62 (2016), no. 4, pp. 2170–2183.

K. Ye and L.-H. Lim, "Schubert varieties and distances between subspaces of different dimensions," SIAM Journal on Matrix Analysis and Applications, (2016), to appear.

K. Ye and L.-H. Lim, "Every matrix is a product of Toeplitz matrices," Foundations of Computational Mathematics, 16 (2016), no. 3, pp. 577–598.

D. Gleich, L.-H. Lim, and Y. Yu, "Multilinear PageRank," SIAM Journal on Matrix Analysis and Applications, 36 (2015), no. 4, pp. 1507–1541.

L.-H. Lim, "Hodge Laplacians on graphs," S. Mukherjee (Ed.), Geometry and Topology in Statistical Inference, Proceedings of Symposia in Applied Mathematics, 73, AMS, Providence, RI, 2015.

A. Rajkumar, S. Ghoshal, L.-H. Lim, and S. Agarwal, "Ranking from stochastic pairwise preferences: recovering Condorcet winners and tournament solution sets at the top," Proceedings of the 32nd International Conference on Machine Learning (ICML), 37 (2015), pp. 665–673.

L.-H. Lim and P. Comon, "Blind multilinear identification," IEEE Transactions on Information Theory, 60 (2014), no. 2, pp. 1260–1280.

T. Schultz, A. Fuster, A. Ghosh, L. Florack, R. Deriche, and L.-H. Lim, "Higher-order tensors in diffusion imaging," pp. 129–161, C.-F. Westin et al. (Eds.), Visualization and Processing of Tensors and Higher Order Descriptors for Multi-Valued Data, Springer-Verlag, Berlin Heidelberg, 2014.

C.J. Hillar and L.-H. Lim, "Most tensor problems are NP-hard," Journal of the ACM, 60 (2013), no. 6, Art. 45, 39 pp.

L.-H. Lim, "Tensors and hypermatrices," Chapter 15, 30 pp., in L. Hogben (Ed.), Handbook of Linear Algebra, 2nd Ed., CRC Press, Boca Raton, FL, 2013.

M. Gu, L.-H. Lim, and C.J. Wu, "PARNES: A rapidly convergent algorithm for accurate recovery of sparse and approximately sparse signals," Numerical Algorithms, 64 (2013), no. 2, pp. 321–347.

**2. New discoveries, inventions, or patent disclosures:**

**Do you have any discoveries, inventions, or patent disclosures to report for this period?**

No

**Please describe and include any notable dates**

**Do you plan to pursue a claim for personal or organizational intellectual property?**

**Changes in research objectives (if any):**

**Change in AFOSR Program Manager, if any:**

**Extensions granted or milestones slipped, if any:**

**AFOSR LRIR Number**

**LRIR Title**

**Reporting Period**

**Laboratory Task Manager**

**Program Officer**

**Research Objectives**

**Technical Summary**

**Funding Summary by Cost Category (by FY, \$K)**

	Starting FY	FY+1	FY+2
Salary			
Equipment/Facilities			
Supplies			
Total			

**Report Document**

**Report Document - Text Analysis**

**Report Document - Text Analysis**

**Appendix Documents**

**2. Thank You**

**E-mail user**

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